

As discussed in the lecture notes, the transport equation for a scalar valued quantity  $h$  on the surface of the unit sphere in an incompressible velocity field  $\mathbf{u}$  can be written in Cartesian coordinates as

$$h_t + \mathbf{u} \cdot (P\nabla h) = 0, \quad (1)$$

where  $\mathbf{u} = [u \ v \ w]^T$  is tangent to the sphere and  $P\nabla$  is the surface gradient. In this problem you will use the differentiation matrices for approximating the surface gradient from problem 4 to solve this equation using the method-of-lines (MOL) technique described in the lecture notes (and in more detail in [1]). We briefly review this below.

Let  $X = \{\mathbf{x}_j\}_{j=1}^N \subset \mathbb{S}^2$  and denote the vectors containing samples of  $h$ ,  $u$ ,  $v$ , and  $w$  at  $X$  as  $\underline{h}$ ,  $\underline{u}$ ,  $\underline{v}$ , and  $\underline{w}$ , respectively. Also let  $D_N^x$ ,  $D_N^y$ , and  $D_N^z$  denote the differentiation matrices associated with the node set  $X$  for the three respective components of the surface gradient. Then, the *semi-discrete* or MOL formulation of (1) is

$$\begin{aligned} \underline{h}_t &= -(\text{diag}(\underline{u})D_N^x + \text{diag}(\underline{v})D_N^y + \text{diag}(\underline{w})D_N^z) \underline{h} \\ &= -D_N \underline{h}. \end{aligned} \quad (2)$$

Given an initial value for  $h$ , this system of ODEs can be numerically solved using any number of numerical methods.<sup>1</sup> For example, one could use the classical 4th order Runge-Kutta scheme (RK4):

$$\begin{aligned} \underline{p}_1 &= -D_N \underline{h}_k \\ \underline{p}_2 &= -D_N \left( \underline{h}_k + \frac{\Delta t}{2} \underline{p}_1 \right) \\ \underline{p}_3 &= -D_N \left( \underline{h}_k + \frac{\Delta t}{2} \underline{p}_2 \right) \\ \underline{p}_4 &= -D_N \left( \underline{h}_k + \Delta t \underline{p}_3 \right) \\ \underline{h}_{k+1} &= \underline{h}_k + \frac{\Delta t}{6} (\underline{p}_1 + \underline{p}_2 + \underline{p}_3 + \underline{p}_4) \end{aligned}$$

where  $\Delta t$  is the time-step,  $\underline{h}_0$  contains the initial condition sampled at  $X$ , and  $\underline{h}_k$  is the numerical solution to (2) at  $k\Delta t$ . This is the numerical scheme you will use in the different problems below.

- 5.a) Following the procedure outlined above, implement a method for solving the transport equation (1) for solid body rotation of a cosine bell, which is Test Case 1 of the standard test suite of Williamson *et. al.* [4]. The velocity field  $\mathbf{u}$  for this test can be computed from the surface curl of the stream function

$$\psi(\mathbf{x}) = \cos(\alpha)z + \sin(\alpha)y,$$

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<sup>1</sup>The method selected should depend on the properties of the eigenvalues of  $D_N$  in order for stability to be maintained. For some velocity fields  $\mathbf{u}$  it may be necessary to add a *hyperviscosity* term to  $D_N$  to stabilize the system as discussed in the lecture notes and problem 5.c)

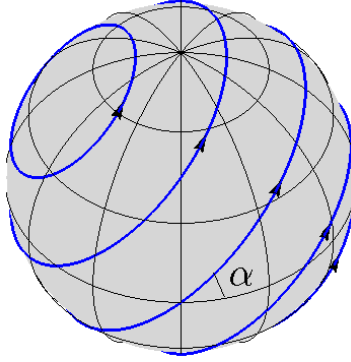


Figure 1: Solid body rotation over the surface of a sphere at an angle  $\alpha$  (blue solid lines) relative to the standard longitude-latitude grid (lighter black solid lines).

where  $\alpha$  corresponds to the angle of the rotation axis as measured from the north pole (with  $\alpha = 0$  being strictly zonal flow from west to east and, and  $\alpha = \pi/2$  being rotation from north to south over the poles); see Figure 1. Computing the surface curl of  $\psi(\mathbf{x})$  gives the velocity field

$$\mathbf{u} = \begin{bmatrix} -y \cos \alpha - z \sin \alpha \\ -x \cos \alpha \\ x \sin \alpha \end{bmatrix}.$$

The initial condition is the compactly supported cosine-bell:

$$h(\mathbf{x}, 0) = \begin{cases} \frac{1}{2} (1 + \cos(3\pi r(\mathbf{x}))) & r(\mathbf{x}) < 1/3, \\ 0 & r(\mathbf{x}) \geq 1/3, \end{cases} \quad (3)$$

where  $r(\mathbf{x}) = \arccos(x)$ , which places the center of the bell at  $(1, 0, 0)$ . Note that this bell is only  $C^1$  on  $\mathbb{S}^2$ . The test is designed so that the bell rotates around the sphere at an angle  $\alpha$  with respect to the polar axis and completes one full revolution at time  $t = 2\pi$ , returning to its initial position. Thus, the exact solution at  $t = 2\pi$  is the same as the initial condition (3), which allows an easy way to check the errors in the numerical solution.

The velocity field and initial condition for this test case can be obtained from the function `setupSBRot` in the `rbfsphere` package.

- (i) Compute the numerical solution over one full revolution (i.e.  $t = 2\pi$ ) for the IMQ kernel using  $N = 1849$  MD nodes,  $\varepsilon = 1.75$ ,  $\alpha = \pi/2$ , and a time-step of  $\Delta t = 240/(2\pi)$ . Plot the numerical solution every 5 time-steps using your functions from problem 1. You should find that there is no visual evidence of a dispersive wave train in the numerical solution and the bell stays sharp and intact even at this low of a resolution. Compute the  $\ell_2$  and  $\ell_\infty$  norm of the difference between the numerical and exact solution at the nodes  $X$ .
  - (ii) Experiment with different  $N$ , node sets, time-steps, rotation angles, and kernels to get a feel for how the method performs.
- 5.b) Adapt your code from 5.a) to solve the idealized cyclogenesis test problem from [3], which specifies a stationary angular velocity field that spins up an initial scalar field, resulting in two diametrically opposed vortices at the poles of the spherical coordinate system. The velocity field for this problem is best expressed with respect to spherical coordinates. Denoting this velocity field as  $\mathbf{u}_s = [u_s \ v_s]^T$ , its components are given by

$$\begin{aligned} u_s &= \omega(\theta) \cos \theta, \\ v_s &= 0 \end{aligned} \quad (4)$$

with

$$\omega(\theta) = \begin{cases} \frac{3\sqrt{3}}{2} \operatorname{sech}^2(\rho(\theta)) \tanh(\rho(\theta)) & \rho(\theta) \neq 0 \\ 0 & \rho(\theta) = 0 \end{cases} \quad (5)$$

where  $\rho(\theta) = \rho_0 \cos(\theta)$  and  $\rho_0$  is a parameter controlling the radial extent of the vortex; in this test  $\rho_0 = 3$ . The initial condition is also best described in spherical coordinates and is given as

$$h(\lambda, \theta, 0) = 1 - \tanh\left(\frac{\rho(\theta)}{5} \sin(\lambda)\right). \quad (6)$$

The analytical solution to (1) at any time  $t$  with this initial condition is given by

$$h(\lambda, \theta, t) = 1 - \tanh\left(\frac{\rho(\theta)}{5} \sin(\lambda - \omega(\theta)t)\right). \quad (7)$$

The test calls for computing the errors in the numerical solution at time  $t = 3$  using the analytical solution (7).

To use the MOL technique described above for this problem, you will need to obtain the velocity field (4) in Cartesian coordinates. This can be done using the the function `setupCyclogenesis` in the `rbfsphere` package, which can also be used to obtain the initial condition. The exact solution (7) can be computed using the function `computeCyclogenesis`.

- (i) Compute the numerical solution up to time  $t = 3$  for the IMQ kernel using  $N = 1849$  MD nodes,  $\varepsilon = 1.75$ , and a time-step of  $\Delta t = 3/24$ . Plot the numerical solution every time-steps using your functions from problem 1 and compare it to the exact solution. Compute the  $\ell_2$  and  $\ell_\infty$  norm of the difference between the numerical and exact solution at the nodes  $X$ .
- (ii) Experiment with different  $N$ , node sets, time-steps, and kernels to get a feel for how the method performs.

5.c) The differentiation matrix  $-D_N$  for the cyclogenesis test case from 5.b) has some eigenvalues with positive real part, meaning that eventually the numeral solution will blow-up. However, the eigenvectors associated with these eigenvalues have high-frequency so that for a smooth initial condition, the blow-up will not be visually detectable immediately in the numerical integration (until  $t \approx 9$  for the initial condition specified above, which is much less than the test case calls for). Nevertheless, this blow-up is troublesome. An effective strategy for stabilizing the method is to include a *hyperviscosity* term in the equations, as discussed in the notes and first proposed by Fornberg and Lehto in [2]. The purpose of this term is to shift the spurious eigenvalues of  $-D_N$  in the right half-plane to the left-half plane without effectively over damping the eigenvalues corresponding to the physically relevant modes in the left-half plane and on the imaginary axis.

For differentiation matrices based on the global RBF method with a positive definite kernel, as used here, Fornberg and Lehto propose using the inverse of the RBF interpolation matrix  $A_N$ , with entires  $(A_N)_{i,j} = \phi(\|\mathbf{x}_i - \mathbf{x}_j\|)$ ,  $i, j = 1, \dots, N$ , for the hyperviscosity term. When the radial kernel  $\phi$  is positive definite  $A_N$  will have all positive eigenvalues so that  $A_N^{-1}$  should be subtracted from  $-D_N$  to shift the spurious eigenvalues to the left half-plane. The modified semi-discrete system (2) with the hyperviscosity term takes for form

$$\underline{h}_t = -(D_N + \mu A_N^{-1}) \underline{h}, \quad (8)$$

where  $\mu > 0$  is a parameter that must be chosen so that the spurious eigenvalues are dampened, but not by too much. Presently there is no theory for choosing this value, although running the code for a few time-steps for different  $\mu$  usually allow one obtain a good value.

- (i) Redo problem 5.b) (i) using hyperviscosity as indicated by (8). Set  $\mu = 5 \cdot 10^{-9}$  and verify that the errors are roughly the same as those from 5.b) (i).
- (ii) Run the simulation up to  $t = 30$  to ensure the solution does not blow-up. Compare this to what happens with no hyperviscosity term.
- (iii) Compare the eigenvalues of  $-D_N$  and  $-(D_N + \mu A_N^{-1})$ .

## References

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