Fast Multilevel Evaluation of 1-D Piecewise Smooth Radial Basis Function Expansions *

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Dedicated to Robert Schumann's Violin Concerto in D Minor, WoO 23 (1853)

Abstract:

Radial basis functions (RBFs) are a powerful tool for interpolating/approximating multidimensional scattered data in $\mathbb{R}^d$. However, a direct evaluation of an n-center RBF expansion at m points requires $O(nm)$ operations, which is prohibitively expensive as n, m increase. We present a new multilevel method for uniformly dense centers and points and $d = 1$, whose cost is only $O(C(n + m))$, where C depends on the desired evaluation accuracy. The method extends a previous work [21] to any piecewise smooth radial kernel, e.g., thin plate spline. The multilevel summation algorithm can be generalized to higher dimensions, and can be also applied beyond RBFs, e.g. to discrete integral transform evaluation and

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Table 1. *Some commonly used radial basis functions. In all cases, $\varepsilon > 0$.*

Type of Radial Kernel	$\phi(r)$
Smooth Kernels	
Gaussian (GA)	$e^{-(\varepsilon r)^2}$
Generalized Multiquadric (GMQ)	$(1+(\varepsilon r)^2)^{\nu/2},\ \nu \neq 0$ and $\nu \notin 2\mathbb{N}$
Piecewise Smooth Kernels	
Generalized Duchon Spline (GDS)	$r^{2k}\log r,\ \ k \in \mathbb{N}$
	$r^{2\nu},\ \ \nu > 0$ and $\nu \notin \mathbb{N}$
Matérn	$\dfrac{2^{1-\nu}}{\Gamma(\nu)} r^{\nu} K_{\nu}(r),\ \ \nu > 0$
Compactly Supported	
Wendland [26]	$(1-r)_+^k p(r),\ \ p =$ polynomial, $k \in \mathbb{N}$

electrostatic force summation.

Key words: Radial basis functions, fast multilevel multi-summation, kernel softening
AMS Classification: *41A21, 41A30, 41A63, 65D25, 65N06, 65R10, 68Q25*

1 Introduction

In many science and engineering disciplines we need to interpolate or approximate a function $f : \mathbb{R}^d \to \mathbb{R},\ d \geq 1$ from a discrete set of scattered samples. Radial basis functions (RBFs) are a simple and powerful technique for solving this problem, with applications to cartography [17], neural networks [16], geophysics [5, 6], pattern recognition [24], graphics and imaging [12, 13, 27], and the numerical solution of partial differential equations [18, 19].

The basic RBF interpolant to a given set of *centers* $\{\mathbf{y}_j\}_{j=0}^n \subset \mathbb{R}^d$ and corresponding *data* $f_j = f(\mathbf{y}_j)$, $j = 0, 1, \ldots, n$, is given by

$$s(\mathbf{x}) = \sum_{j=0}^{n} \lambda(\mathbf{y}_j)\phi\left(\|\mathbf{x} - \mathbf{y}_j\|_2\right)\ , \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^d$, ϕ is a univariate, *radially symmetric kernel*, and the *expansion coefficients* $\{\lambda(\mathbf{y}_j)\}_{j=0}^n$ are determined by the linear system of equations $s(\mathbf{y}_j) = f_j$, $j = 0, 1, \ldots, n$. The well-posedness of (1) and its generalization to approximation, instead of interpolation, is discussed in [11]. Some common choices of ϕ are listed in Table 1.

While RBFs are a powerful, mathematically elegant technique for approximating function and derivatives in multiple dimensions, their use in large scale computation has been hindered by two primary computational challenges:

(i) *Fitting:* Given $\{\mathbf{y}_j\}_{j=0}^n$ and $\{f_j\}_{j=0}^n$, determine the expansion coefficients $\{\lambda(\mathbf{y}_j)\}_{j=0}^n$. For most radial kernels, this problem gives rise to $(n+1)$-by-$(n+1)$ *dense* linear system of equations which, if solved directly, requires $O(n^3)$ operations.

(ii) *Evaluation:* Given $\{\mathbf{y}_j\}_{j=0}^n$ and $\{\lambda(\mathbf{y}_j)\}_{j=0}^n$, evaluate the RBF interpolant (1) at multiple points $\mathbf{x} = \mathbf{x}_i, \ \ i = 0, 1, \ldots, m$. For most radial kernels, this problem gives rise to a *multi-summation* task which, if directly evaluated, requires $O(mn)$ operations.

Krylov-subspace iterative methods can be used to speed-up the computation of (i) [3, 14, 15]. However, these methods require the ability to efficiently evaluate the RBF interpolant (i.e. compute matrix-vector products). Therefore, overcoming (ii) is fundamental to overcoming (i).

For *smooth* kernels (e.g., Table 1, top section), we presented a fast multilevel algorithm in [21] for reducing the evaluation of (1) at $m+1$ points $\mathbf{x}$ to $O((\log(1/\delta))^d(n+m))$ operations, where δ is the desired evaluation accuracy. In this paper, we generalize this algorithm to the case of a 1-D ($d = 1$) evaluation with *piecewise smooth* radial kernels ϕ (e.g., Table 1, middle section) and uniformly dense centers and evaluation points. Namely, given $\{y_j\}_{j=0}^n \subset \mathbb{R}$, $\{x_i\}_{i=0}^m \subset \mathbb{R}$, and $\lambda(y_j)$, compute

$$s(x_i) = \sum_{j=0}^{n} \lambda(y_j)\phi\left(|x_i - y_j|\right) \ , \quad i = 0, 1, \ldots, m \ . \tag{2}$$

As in the smooth kernel case, our algorithm reduces the $O(mn)$ computational cost of (2) to $O(\log(1/\delta)(m+n))$, where δ is the desired evaluation accuracy. The generalization of our algorithm to higher dimensions and non-uniform densities will be presented in a future paper as discussed in §4.

For smooth kernels, our multilevel algorithm [21] required only two levels to achieve a linearly scaling evaluation cost. For piecewise smooth kernels, we have to construct a multilevel hierarchy of successively coarser levels from the original centers $\{y_j\}_{j=0}^n$ and evaluation points $\{x_i\}_{i=0}^m$ due to the singularity of the derivatives of ϕ at $r = 0$. The original multilevel approach we build on was developed by Brandt for evaluating discrete integral transforms and electrostatic particle interactions [7].

Alternative fast RBF expansion methods are the *Fast Multipole Method (FMM)* [4] and the *Fast Gauss Transform (FGT)* [22, 23]. The main advantages of our method are its relatively simple implementation, easy parallelization, large scope of application to any piecewise smooth kernel in any dimension, and precise analysis of evaluation error and complexity. For further discussion comparing our multilevel approach with these methods, see [21, §1].

The paper is organized as follows: In §2 we introduce kernel softening, an important tool in the multilevel evaluation algorithm. In §3 we describe our fast evaluation algorithm. Numerical results for specific piecewise smooth kernels are presented in §3.4. We discuss generalizations and future research in §4.

2 Kernel Softening

Piecewise smooth kernels are increasingly smoother for larger r, but have a singularity in one of their derivatives at $r = 0$. A special case of interest is the popular thin plate spline kernel, $\phi(r) = r^2 \log(r)$, which has a jump in the second derivative at $r = 0$. Consequently, we cannot directly apply the two-level evaluation algorithm of [21], because the interpolation errors would be uncontrolled over a large neighborhood of $r = 0$. To address this difficulty, the kernel is decomposed into two parts [7, 8, 10],

$$\phi(r) = \phi_A(r) + \phi_{\mathrm{local,A}}(r), \tag{3}$$

so that

(i) $\phi_A(r) = \phi(r)$ (or $\phi_{\mathrm{local,A}}(r) = 0$) for all $|r| \geq A$.

(ii) ϕ_A is *scale-A-smooth*, namely, for any $\delta > 0$ there exists $p = O(\log(1/\delta)) \in \mathbb{N}$ such that ϕ_A can be uniformly approximated to δ-accuracy by a pth-order polynomial interpolation from its values on any uniform grid with a mesh size comparable with A [10, 20].

We use a polynomial softened kernel ϕ_A that interpolates $\phi, \phi', \ldots, \phi^{(p)}$ at $r = A$ and has vanishing odd derivatives up to order p at $r = 0$ (as required for a smooth radially symmetric function). To satisfy these conditions, we assume ϕ_A has the form (see also [7, 10])

$$\phi_A(r) = \begin{cases} \phi^*\left(\frac{r}{A}\right), & |r| \leq A, \\ \phi(r), & |r| \geq A, \end{cases} \qquad \phi^*(r) := \sum_{k=0}^{p} a_k \left(r^2 - 1\right)^k. \tag{4}$$

The solution to this generalized Hermite interpolation can be computed from the pth order Taylor expansion of $\phi(A\sqrt{t})$ about $t = 1$ [2, p. 163]. Thus, for many kernels ϕ, (4) can be analytically computed for any p. For example, for the thin plate spline

$$\phi^*(r) = A^2 \log(A) + A^2 \left(\frac{1}{2} + \log(A)\right)(r^2 - 1) + \sum_{k=2}^{p} \frac{(-1)^k A^2}{2k(k-1)}(r^2 - 1)^k. \tag{5}$$

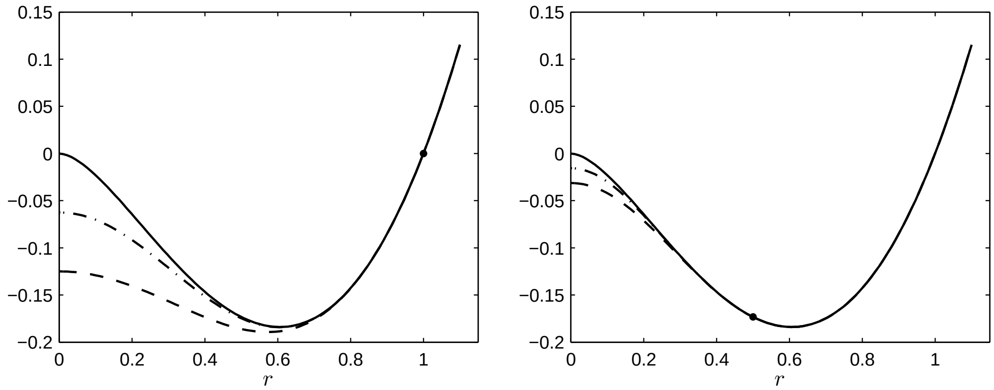
Figure 1 illustrates the softening of this kernel. An alternative kernel softening technique that minimizes the derivative amplitudes of ϕ_A is described in [20].

The relative error $\delta_{\mathcal{I}}$ in approximating the scale-A softened kernel $\phi_A(r)$ by a centered, even, pth-order interpolation from its values on a mesh size-H uniform grid is bounded by [25, p. 32]

$$\delta_{\mathcal{I}} = \left| \phi\left(|x_i - y_j|\right) - \sum_{J \in \sigma_j} \omega_{jJ} \phi_A\left(|x_i - Y_J|\right) \right| \leq \frac{H^p \left[\Gamma\left(\frac{p+1}{2}\right)\right]^2}{p!\pi} \left| \phi_A^{(p)}(x_i - \xi) \right|, \tag{6}$$

where ξ is in the convex hull of $\{Y_J\}_{J \in \sigma_j}$, $j = 0, 1, \ldots, n$. For the thin plate spline, it can be shown that $|\phi_A^{(p)}|$ attains it maximum at $r = 0$. Thus,

$$\left\| \phi_A^{(p)} \right\|_\infty \leq \frac{1}{A^{p-2}} \frac{4^p \left[\Gamma\left(\left(\frac{p+1}{2}\right)\right]^2}{\pi p(p-2)}. \tag{7}$$



(a) (b)

Figure 1. *The kernel $\phi(r) = r^2 \log(r)$ (solid line) and the softened kernel $\phi_A(r)$ for $p = 4$ (dashed line) and $p = 8$ (dashed-dotted line) for (a) $A = 1$ and (b) $A = \frac{1}{2}$.*

Using this bound, (6), and Stirling's asymptotic formula [1, p. 257], we get

$$\delta_{\mathcal{I}} \lesssim \sqrt{\frac{2}{\pi}} \frac{2A^2}{p^{3/2}(p-2)} \left(\frac{pH}{eA}\right)^p \tag{8}$$

For other piecewise smooth kernels, we get a similar bound (see for example [20, App. A]). In fact, numerical experiments suggest that for many kernels the maximum of $|\phi_A^{(p)}|$ is attained at $r = 0$, which simplifies the analysis.

3 1-D Fast Evaluation Algorithm

For simplicity, we describe the algorithm when the centers $\{y_j\}_{j=0}^n$ and evaluation points $\{x_i\}_{i=0}^m$ have the same average density h. Furthermore, without loss of generality, we assume $\{x_i\}_{i=0}^m, \{y_j\}_{j=0}^n \subseteq [0, 1]$.

3.1 The Two-Level Case

The idea of the algorithm is to write the RBF expansion as

$$s(x_i) = s_A(x_i) + s_{\text{local,A}}(x_i), \quad i = 0, 1, \ldots, m; \tag{9}$$

$$s_A(x_i) = \sum_{j=0}^{n} \lambda(y_j) \phi_A\left(|x_i - y_j|\right), \quad i = 0, 1, \ldots, m; \tag{10}$$

$$s_{\text{local,A}}(x_i) = \sum_{j:|x_i - y_j| < A} \lambda(y_j) \phi_{\text{local,A}}\left(|x_i - y_j|\right), \quad i = 0, 1, \ldots, m. \tag{11}$$

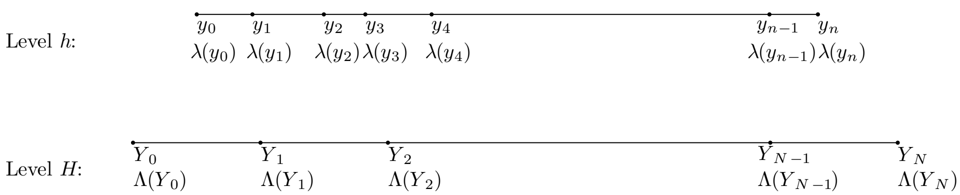


Figure 2. *Illustration of the two-level approach for the case* $p = 2$. $\{X_I\}_{I=0}^{M}$ *are similarly defined over* $\{x_i\}_{i=0}^{m}$. *Level* H *contains* $O(p)$ *points outside the convex hull of level* h *to allow centered interpolation.*

Since ϕ_A is a smooth kernel, (10) is evaluated using the two-level fast evaluation method of [21]. Since $\phi_{\mathrm{local,A}}$ is *compactly supported*, (11) is evaluated directly.

We define an auxiliary level $H = 2h$ consisting of two uniform grids $\{Y_J\}_{J=0}^{N}$ and $\{X_I\}_{I=0}^{M}$ with spacing H each, and a comparable softening distance $A = aH, a = O(1)$, so that ϕ_A is smooth at scale H. The H-grids cover $\{y_j\}_{j=0}^{n}$ and $\{x_i\}_{i=0}^{m}$, respectively, so that a discrete function defined at level H can be approximated at any y_j using *centered* pth-order interpolation, for some even positive integer p (see Figure 2 for an illustration of the two-levels). The evaluation algorithm replaces the expensive summation (10) at level h by a less expensive summation at level H by utilizing the spatial smoothness of ϕ_A.

The first step is to note that $\phi_A(|x_i - y|)$ is a smooth function of y at scale H. Therefore its value at $y = y_j$ can be approximated by a centered pth-order interpolation from its values at neighboring Y_J's. Namely,

$$\phi_A\left(|x_i - y_j|\right) = \sum_{J \in \sigma_j} \omega_{jJ} \phi_A\left(|x_i - Y_J|\right) \, + O(\delta_{\mathcal{I}}), \quad j = 0, 1, \ldots, n\,, \tag{12}$$

where $\sigma_j := \{J : |Y_J - y_j| < pH/2\}$, ω_{jJ} are the centered pth-order interpolation weights from the coarse centers Y_J to y_j, and $\delta_{\mathcal{I}}$ is the interpolation error, bounded by (8).

Substituting the approximation (12) into (10) and interchanging the order of summation, we obtain

$$\begin{aligned} s_A(x_i) &= \sum_{j=0}^{n} \left[\sum_{J \in \sigma_j} \omega_{jJ} \phi_A\left(|x_i - Y_J|\right) \, + O(\delta_{\mathcal{I}}) \right] \lambda(y_j) \\ &= \sum_{J=0}^{N} \phi_A\left(|x_i - Y_J|\right) \sum_{j : J \in \sigma_j} \omega_{jJ} \lambda(y_j) \, + O(n \|\boldsymbol{\lambda}\|_\infty \delta_{\mathcal{I}}) \\ &= \sum_{J=0}^{N} \Lambda(Y_J) \phi_A\left(|x_i - Y_J|\right) \, + O(n \|\boldsymbol{\lambda}\|_\infty \delta_{\mathcal{I}}), \quad i = 0, 1, \ldots, m\,, \end{aligned} \tag{13}$$

where

$$\Lambda(Y_J) := \sum_{j : J \in \sigma_j} \omega_{jJ} \lambda(y_j), \; J = 0, 1, \ldots, N, \tag{14}$$

which is called *anterpolation* [7] or aggregation of $\{\lambda(y_j)\}_{j=0}^{n}$ to level H. The upper bound for the total interpolation error in $s_A(x_i)$ is $O(n\|\boldsymbol{\lambda}\|_\infty \delta_{\mathcal{I}})$, where $\|\boldsymbol{\lambda}\|_\infty$ is the maximum norm of $\{\lambda(y_j)\}_{j=0}^{n}$. In fact, if we assume that local errors accumulate randomly, the total error will only be $O(\sqrt{n}\|\boldsymbol{\lambda}\|_\infty \delta_{\mathcal{I}})$.

Second, we similarly use the smoothness of $\phi_A(|x - Y_J|)$ as a function of x for a fixed Y_J to approximate its value at $x = x_i$ by a centered pth-order interpolation from its values neighboring X_I's. Namely,

$$\phi_A\left(|x_i - Y_J|\right) = \sum_{I \in \overline{\sigma}_i} \overline{\omega}_{iI} \phi_A\left(|X_I - Y_J|\right) \; + O(\delta_{\mathcal{I}}), \quad i = 0, 1, \ldots, m\,. \tag{15}$$

where $\overline{\sigma}_i := \{I : |X_I - x_i| < pH/2\}$, and $\overline{\omega}_{iI}$ are the centered pth-order interpolation weights from the coarse evaluation points X_I to x_i. Substituting (15) into (13) gives

$$\begin{aligned} s_A(x_i) &= \sum_{J=0}^{N} \Lambda(Y_J) \left[\sum_{I \in \overline{\sigma}_i} \overline{\omega}_{iI} \phi_A\left(|X_I - Y_J|\right) \; + O(\delta_{\mathcal{I}}) \right] + O(n\|\boldsymbol{\lambda}\|_\infty \delta_{\mathcal{I}}) \\ &= \sum_{I \in \overline{\sigma}_i} \overline{\omega}_{iI} \sum_{J=0}^{N} \Lambda(Y_J) \phi_A\left(|X_I - Y_J|\right) \; + O(n\|\boldsymbol{\lambda}\|_\infty \delta_{\mathcal{I}}) \\ &= \sum_{I \in \overline{\sigma}_i} \overline{\omega}_{iI} S_A(X_I) \; + O(n\|\boldsymbol{\lambda}\|_\infty \delta_{\mathcal{I}}), \quad i = 0, 1, \ldots, m\,, \end{aligned} \tag{16}$$

where

$$S_A(X_I) := \sum_{J=0}^{N} \Lambda(Y_J) \phi_A\left(|X_I - Y_J|\right), \; I = 0, 1, \ldots, M\,. \tag{17}$$

Thus, the original evaluation task (10) is replaced by the less expensive, analogous evaluation task (17) at level H. The final step in computing $\{s(x_i)\}_{i=0}^{m}$ in (2) is to combine (16) with the local correction (11).

We note that if $\{y_j\}_{j=0}^{n}$ and $\{x_i\}_{i=0}^{m}$ have different uniform densities (e.g. $h(y)$ and $h(x)$, respectively), then the only change to the algorithm is that the corresponding coarse levels $\{Y_J\}_{J=0}^{N}$ and $\{X_I\}_{I=0}^{M}$ have different mesh sizes (e.g. $H(Y) = 2h(y)$ and $H(X) = 2h(x)$, respectively).

3.2 The Multilevel Case

The number of nodes at level H is roughly $(n+m)/2$, which may still be too large to sum directly. Instead, we apply the same two-level algorithm: ϕ_A is softened to ϕ_{2A} plus a local part; the former is computed using a yet coarser grid $(2H)$ and the latter is directly evaluated. In general, we use $L \sim \log(n+m)/2$ coarser levels below the original level h, until a level is reached at which the analog of (17) can be directly evaluated in $O(n+m)$ operations. Figure 3 illustrates this recursive procedure for $L = 2$.

Our fast evaluation task is summarized in the following recursive algorithm:

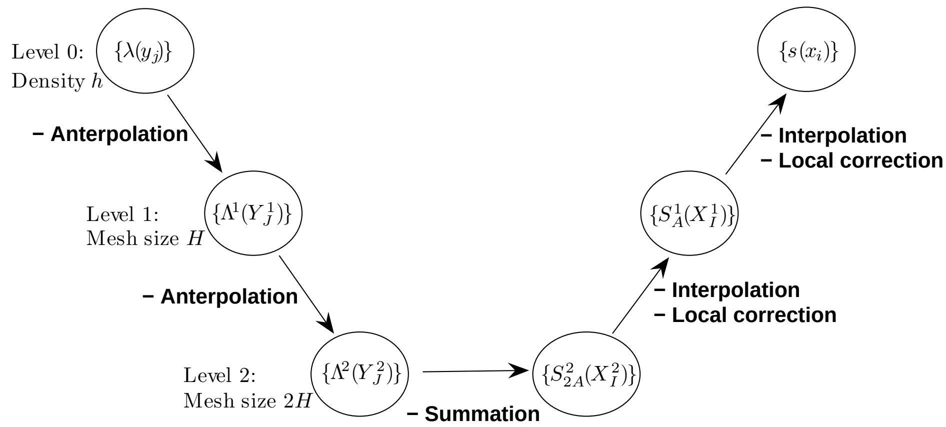


Figure 3. *Illustration of the recursive multilevel algorithm for the case of three levels ($L = 2$). Capital letters denote coarse level quantities; superscript l denotes a quantity at level l. "Level 0" denotes the original points and centers.*

Algorithm 3.1 $\{s(x_i)\}_{i=0}^m = \mathrm{EVAL}(\phi,\, h,\, p,\, a,\, \{x_i\}_{i=0}^m,\, \{y_j\}_{j=0}^n,\, \{\lambda(y_j)\}_{j=0}^n)$

1. *Anterpolation:*
   - (i) Define $H = 2h, A = aH$.
   - (ii) For $j = 0, 1, \ldots, n$, compute the anterpolation weights $\{\omega_{jJ}\}_{J\in\sigma_j}$.
   - (iii) Compute the coarse expansion coefficients $\{\Lambda(Y_J)\}_{J=0}^N$ using (14).
2. *Coarse Level Summation:*
   - (i) If $N + M = O(\sqrt{n+m})$ then directly evaluate $S_A(X_I)$, $I = 0, 1, \ldots, M$.
   - (ii) Otherwise, evaluate on the next coarsest level: $\{S_A(X_I)\}_{I=0}^M = \mathrm{EVAL}(\phi_A,\, H,\, p,\, a,\, \{X_I\}_{I=0}^M,\, \{Y_J\}_{J=0}^N,\, \{\Lambda(Y_J)\}_{J=0}^N)$.
3. *Interpolation:*
   - (i) For $i = 0, 1, \ldots, m$, compute the interpolation weights $\{\overline{\omega}_{iI}\}_{I\in\overline{\sigma}_i}$.
   - (ii) Interpolate $\{S_A(X_I)\}_{I=0}^M$ to $\{s_A(x_i)\}_{i=0}^m$ using (16).
4. *Local correction:*
   - (i) For $i = 0, 1, \ldots, m$, compute the local sum $\{s_{\text{local,A}}(x_i)\}_{i=0}^m$ in (11).
   - (ii) $\{s(x_i)\}_{i=0}^m = \{s_A(x_i)\}_{i=0}^m + \{s_{\text{local,A}}(x_i)\}_{i=0}^m$.

Return $\{s(x_i)\}_{i=0}^m$.

3.3 Complexity and Accuracy

We only give a general outline here; the full analysis will appear in a future work. We first analyze the two-level case. Step 1 consists of two parts. Computing the weights $\{\omega_{jJ}\}_J$ requires $O(np)$ operations (see [21, App. A]); then (14) is executed in $O(np)$ operations [21, §2.2]. Step 3 consists of computing $\{\overline{\omega}_{iI}\}_I$, which costs $O(mp)$, and interpolating the level H RBF expansion to level h for a cost of $O(mp)$. Step 5 costs $O(ma)$ operations, because there are m terms, each consisting of $O(a)$ per our assumption of uniformly dense centers and points. Hence, the complexity is W is $O((n+m)p+ma)$ (when step 2 is neglected).

Let $\mathbf{s}$ contain the values from directly summing (2) and $\hat{\mathbf{s}}$ the contain the values from our fast evaluation for $i = 0, 1, \ldots, m$. We define the evaluation accuracy as the relative error norm

$$E := \frac{\|\mathbf{s} - \hat{\mathbf{s}}\|_\infty}{\|\mathbf{s}\|_\infty}. \tag{18}$$

Using (8) (or a similar bound for other kernels) for $A = aH, H = 2h$, we get

$$\delta_{\mathcal{I}} \lesssim (aH)^2 \left(\frac{kp}{a}\right)^p, \tag{19}$$

for some constant $k = O(1)$. The same bound applies to E (assuming the evaluation problem (2) is well-conditioned). The optimal parameters that minimize W subject to a prescribed error tolerance $E \le \delta$ can be shown to be

$$p_{\text{opt}}(\delta) = \log\left(\frac{1-c}{c}\right)\log\left(\frac{H^2}{\delta}\right), \tag{20}$$

$$a_{\text{opt}}(\delta) = \frac{1-c}{c}\log\left(\frac{1-c}{c}\right)\log\left(\frac{H^2}{\delta}\right), \tag{21}$$

where c is some $O(1)$ constant. In practice, p is rounded to the next even integer and a is rounded to the next integer.

Using the optimal parameters we obtain

$$W \sim O\left((n+m)p_{\text{opt}}(\delta) + ma_{\text{opt}}(\delta)\right) \sim (n+m)\log\left(\frac{H^2}{\delta}\right) \lesssim (n+m)\log\left(\frac{1}{\delta}\right) \tag{22}$$

(note that $H \le 1$). In the case of L coarse levels, the total error is at worst the sum of all errors accumulated in L coarsening stages. Hence, to obtain an error below a certain δ, we use

$$p_l = p_{\text{opt}}(2^{-l-1}\delta), \qquad a_l = a_{\text{opt}}(2^{-l-1}\delta) \tag{23}$$

when calling EVAL at level l, $l = 0, \ldots, L-1$. This yields

$$E \lesssim \sum_{l=0}^{L-1} 2^{-l-1}\delta \le \delta.$$

n	$\delta = 10^{-2}$	$\delta = 10^{-4}$	$\delta = 10^{-6}$	$\delta = 10^{-7}$	$\delta = 10^{-8}$
64	$2.83 \cdot 10^{-3}$	$9.03 \cdot 10^{-5}$	$2.68 \cdot 10^{-8}$	$8.93 \cdot 10^{-10}$	$2.87 \cdot 10^{-11}$
256	$9.83 \cdot 10^{-4}$	$3.25 \cdot 10^{-5}$	$2.07 \cdot 10^{-7}$	$7.18 \cdot 10^{-9}$	$6.61 \cdot 10^{-11}$
1024	$3.25 \cdot 10^{-4}$	$1.15 \cdot 10^{-5}$	$3.10 \cdot 10^{-7}$	$1.12 \cdot 10^{-8}$	$5.16 \cdot 10^{-10}$
4096	$7.65 \cdot 10^{-5}$	$6.40 \cdot 10^{-6}$	$2.22 \cdot 10^{-7}$	$1.63 \cdot 10^{-8}$	$7.93 \cdot 10^{-10}$

Table 2. *Relative ℓ_∞ error of the multilevel evaluation method for the thin plate spline, versus n and δ ($n = m$).*

The total work in the multilevel algorithm is then

$$\begin{aligned} W &\sim \sum_{l=0}^{L-1} 2^{-l} \left((n+m)p_l + ma_l\right) \sim (n+m) \sum_{l=0}^{L-1} 2^{1-l} \log\left(\frac{2^{l+1}}{\delta}\right) \\ &\leq \left(2 + \log(2) + 2\log(\frac{1}{\delta})\right)(n+m) \\ &\sim (n+m)\log\left(\frac{1}{\delta}\right). \end{aligned} \tag{24}$$

To sum up, choosing the algorithm's parameters at all levels using (23) with (21), the algorithm computes (2) to δ-accuracy in $W \sim O(\log(1/\delta)(n+m))$ operations.

3.4 Numerical Experiments

We numerically verify the accuracy and work estimates for our multilevel method derived above for the case of $\phi(r) = r^2 \log(r)$. The interpolation orders and softening distances at all levels were chosen using (23) with (21) and a non-optimized $c = 0.25$.

First, we verify that with this choice of parameters the relative error E (18) is indeed $O(\delta)$. Table 2 shows the E for various values of n and δ. Each entry in the table is the average of ten different experiments, where $\{y_j\}_{j=0}^n$ and $\{x_i\}_{i=0}^m$ were sampled from a uniform random distribution of $[0,1]$, and $\{\lambda(y_j)\}_{j=0}^n$ were sampled from a normal random distribution of $[-1,1]$ in each experiment. We see that E is below δ in all cases.

Second, we verify that the work W (24) linearly scales with m, n, and $\log(1/\delta)$. Figure 4 compares the number of operations required for our multilevel method for various values of n and δ, with a direct evaluation. Each evaluation of ϕ or ϕ_A is counted as one operation. As expected, the results follow the work estimate (24).

4 Concluding Remarks

We presented a fast RBF evaluation algorithm for piecewise smooth radial kernels in 1-D. The algorithm scales linearly with the number of centers and evaluation points. Numerical results with the popular thin-plate spline kernel confirm the theoretical accuracy and work estimates. This fast evaluation will hopefully provide

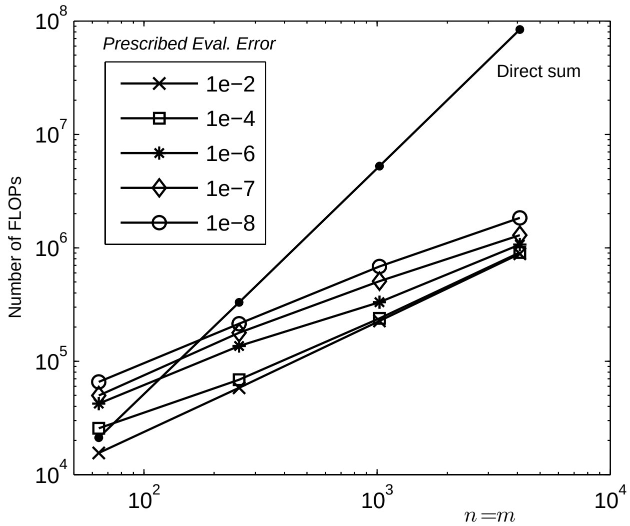


Figure 4. *Comparison of the number of floating point operations (FLOPs) required for the multilevel algorithm vs. a direct evaluation with the thin plate spline kernel. The n centers and $m = n$ evaluation points were sampled from a uniform random distribution of* $[0, 1]$.

an important tool to be integrated into existing RBF interpolation/approximation software, and will allow faster solutions of large-scale interpolation problems. The algorithm can be efficiently *parallelized*, as explained in [21]. Other directions for future research follow.

Higher dimensions. The algorithm can be extended to any dimension d, using a hierarchy of uniform coarse grids in $\mathbb{R}^d$, and tensor products of 1-D interpolations and anterpolations, as in [21, §3]. The cost of the evaluation is then $O((\log(1/\delta))^d(n + m))$ operations.

Non-uniform density. When the density h of the original centers (and/or evaluation points) varies, the coarsening strategy must be modified to maintain optimal evaluation complexity. A convenient option is to still use uniform coarse grids H, but have the first few ones extend only over the high-density parts of the domain. The rest of the algorithm remains the same. This leads to an Adaptive Mesh Refinement (AMR) hierarchy, as explained in [20, §4].

Fast Fitting. The fast multilevel evaluation can compute matrix-vector multiplication in $O(n)$ operations. It can be integrated into any of the existing iterative methods such as Krylov-subspace method [3, 14, 15] for computing $\{\lambda(y_j)\}_{j=0}^n$ from given data $\{f_j\}_{j=0}^n$. Moreover, for some piecewise smooth kernels (e.g. GDS), the entire fitting problem can be solved by a multilevel "V-cycle" solver, along the lines of [9]. The cost of solving the fitting problem to a reasonable tolerance (analogous to the truncation error in discretizing a continuous integral transforms at the centers) is then estimated to be $2 - 3$ matrix-vector multiplications.

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